18-642: Floating Point Blues
10/17/2019

Only two things are infinite, the universe and human stupidity, and I'm not sure about the former.

— Albert Einstein —

http://www.azquotes.com/quote/87292
Floating Point Math

Anti-Patterns:
- Not accounting for roundoff errors
  - Tests for floating point equality
- Not handling special values
- Float used if integer does the job
  - Not always good for “big” numbers

Floating Point Math:
- Exponent + Mantissa representation
  - 32-bit, 64-bit, others on some systems
- Roundoff errors due to finite number of mantissa bits
- Special values:
  - Infinity, Not A Number (NaN), denorms, signed zero

IEEE Floating Point Format
Single Precision: 32 bits total

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<tr>
<th>S</th>
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<th>MANTISSA</th>
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<tbody>
<tr>
<td>1 bit</td>
<td>23 bits (with implicit leading 1.)</td>
<td></td>
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Value = (+/-) 1.Mantissa * 2^(Exponent-127)

Sign: 0=positive; 1=negative
Exponent: 127 bias, radix 2
  value is EXPONENT – 127
Mantissa: implicit 1.
  value is 1.MANTISSA (binary)
Special zero value:
  zero = 0x00000000
Special Values

- **Inf: Infinity**
  - E.g., result when dividing by zero, or overflow

- **Denormalized**
  - Number smaller than smallest fraction
    - $\sim10^{-45} \ldots \sim10^{-38}$ No implicit leading 1 in mantissa

- **NaN: “Not a Number”**
  - E.g., square root of negative number
  - Signaling NaN throws exception
  - Default is usually “silent” NaN (no exception)

- **Silent NaN Comparison Pitfall:**
  - Comparison with NaN is always false
  - if (CurrentSpeed > SpeedLimit) {shutdown}
    - Comparison is false for CurrentSpeed of NaN $\Rightarrow$ no shutdown
  - (NaN == NaN) is also false (surprise!); use isnan()

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Exponent indicates special values:
- zero: 0x80000000
- zero: 0x00000000
- one: 0x3F800000
- +infinity: 0x7F800000
- +NaN Signaling: 0x7F800001...
- +NAN Quiet: 0x7FC00000...
- ...
NaN and the Robot Apocalypse

RECbot Speed Limit Tests

- cmd = 1 m/s: No speed limit violation
- cmd = 3 m/s: Speed limit enforced
- cmd = Inf: Speed limit enforced
- cmd = NaN: Speed limit violated

End of test

Speed-limit violation occurred when exceptional input sent as speed command.
Rounding Errors

Rounding error due to limited bits
- Mantissa: 24 bits (implicit leading one)
  - E.g.: all zero mantissa bits $\rightarrow 1.000000000000000000000002$
- More than 24 bits of value won’t fit
  - Converting int to float to int to float in a chain gives:
    $0x72345673 \rightarrow 1916032640.0 \rightarrow 0x72345680 \rightarrow 1916032640.0$

Rounding error due to imprecise representation
- IEEE 754 is radix 2, so decimal fractions can be inexact
  - Repeatedly add 0.1 to a 32-bit float and you get…
    $0.1, 0.2, \ldots, 2.799999, \ldots, 49.999809, \ldots, 99.999046$

Floating point comparison pitfall:
- if (fa == fb) might not match due to rounding error
  - Use a comparison interval, e.g.,
    if (fabs(result - expectedResult) < 0.00001)
Prove you are human:

0.1 + 0.2 = ?

WELCOME TO THE SECRET ROBOT INTERNET

0.30000000000000004
Don’t Use Floats for Time!

- **Patriot Missile mishap**
  - 1991: Scud kills 28 American (Desert Storm)
  - [http://www.fas.org/spp/starwars/gao/im92026.htm](http://www.fas.org/spp/starwars/gao/im92026.htm)
    - “after about 20 hours, the inaccurate time calculation becomes sufficiently large to cause the radar to look in the wrong place”
      - “Range gate” used to look where target is predicted to be next
      - Target track is lost if range gate is wrong, resulting in a miss
      - The incident happened 100 hours after the last system reset

- **What was the root cause?**
  - Patriot designed for aircraft and frequent mobile relocations
    - Scud missiles travel at Mach 5 (3750 mph); Patriot deployed in fixed location
  - Even a small round-off error matters when computing distance $= \text{velocity} \times \text{time}$
    - Large accumulated base time and high velocity leads to a failure
Time is integer 10ths of second
- Converted to 24-bit fractional value for calculation
- 0.1 seconds is not an "even number" = 0.00011001100110011001100110011001...
- At 100 hours, resultant round-off is 0.000000095 decimal  [https://goo.gl/5ik1au]

After 100 hours error was 0.344 seconds = 697 meters error (per GAO report)
Best Practices for Floating Point

- Use integer math if you can
  - Scaled integer (e.g., 10ths of a second)
  - Binary Coded Decimal (BCD) + radix point
  - Fixed point (e.g., value *256)

- Handle special values
  - NaN is especially tricky to get right

- Manage and handle roundoff error
  - Doubles give more bits to work with (53-bit mantissa)
    - But fundamentally, all problems are still there
  - Don’t use floating point as an iterator, including time!

- Comparisons are especially problematic (NaN, roundoff)
Hey, check it out: $e^\pi - \pi$ is 19.999099979. That's weird.

Yeah, that's how I got kicked out of the ACM in college.

What?

During a competition, I told the programmers on our team that $e^\pi - \pi$ was a standard test of floating-point handlers -- it would come out to 20 unless they had rounding errors.

That's awful.

Yeah, they dug through half their algorithms looking for the bug before they figured it out.

https://xkcd.com/217/